On the Geometry of Constant Returns

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Abstract

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Constant returns to scale, always a simplifying assumption, is often also much more: many important results depend critically on the very special properties of this class of production function. This paper provides a unified set of simple proofs for most of the crucial analytical properties of constant returns production and their implications for firm costs. Only familiar diagrams and high school geometry are used, and the proofs are written to be easily accessible to college sophomores.

1 Introduction

Students are typically introduced to constant returns production in intermediate microeconomics. Thereafter—in courses on international trade, industrial organization, public finance, and other applied fields—they will need to have mastered the special properties of constant returns production if they are to fully understand the many telling illustrations, useful examples, and sometimes central results that ultimately depend upon them.

In the theory of international trade, the Heckscher-Ohlin Theorem, along with its principal corollaries and extensions, including the Factor Price Equalization Theorem, depend crucially on the special properties of constant returns production. So, too, does the Non-Substitution Theorem in general equilibrium, and the Product Exhaustion Theorem in the theory of distribution. In the theory of economic growth, constant returns is an indispensable assumption in neoclassical (Solow) growth models, as well as many newer two-sector endogenous growth models. Ubiquitous in theory and applications, constant returns is usually far from a benign assumption–instead, it is usually at the very heart of the chain of reasoning and the eventual result.

When production displays constant returns to scale, firm cost functions take simple and analytically convenient forms. With two factors and fixed factor prices, short-run average cost is U-shaped and short-run marginal cost is upward sloping; in the long run, average and marginal cost are constant and equal to one another. This combination of conventional short-run and simple long-run cost behavior accounts for much of the popularity of constant returns with theorists and teachers.

Because it typically precedes the specialized field courses, the intermediate micro class is a good place to become acquainted with the unique properties of constant returns. Yet many instructors choose not to employ calculus at this level of the curriculum, and so may feel they must simply assert without proof what the student will need to know. Fortunately, this is not the case.

This paper provides simple proofs for most of the crucial analytical properties of constant returns production and their implications for firm costs. Only familiar diagrams and high school geometry are used, and the proofs are written to be easily accessible to college sophomores. Technical jargon that will normally be unfamiliar to students has been minimized so that students may read and fully comprehend the arguments in this paper.

2 Geometry

Consider a typical production function, f, summarizing efficient possibilities for combining two inputs to produce a single output. Let

$$Q = f(L, K)$$

denote the greatest quantity of output that can be produced during some period of time if the firm uses amounts L and K of two factors we'll call labor and capital, respectively. We will assume throughout that $f(0, 0) = 0$ and that this production function is strictly increasing in non-negative L and K, so that (1) no output is possible without positive amounts of some input, and (2) the marginal product of labor, $MPL \equiv \Delta Q/\Delta L$, and the marginal product of capital, $MPK \equiv \Delta Q/\Delta K$, are finite and strictly positive everywhere.

A production function such as this can be represented by its isoquant map. Each isoquant is a level-curve of the production function drawn in the (L, K)-plane, giving all combinations of the inputs capable of producing a common level of output. Under our assumptions so far, some isoquant will pass through every point in the (L, K)-plane, isoquants will not cross, they will be negatively sloped, and isoquants denoting greater levels of output will lie farther from the origin in a northeasterly direction.

With any production function, it is useful to distinguish between two broad classes of properties: these are its *returns to variable proportions* and its *returns to scale*. When we consider returns to variable proportions, we ask how output behaves as more or less of a variable factor is combined with a given amount of some fixed factor, and this is most relevant to the firm's decision making in the short run. When we consider returns to scale, we hold constant the proportions in which factors are combined, and ask how output behaves as the scale with which both are employed is increased or decreased together. Returns to scale are thus most relevant to the firm's long-run production decisions because only in the long run is the firm free to vary all the factors it uses.

Set Figure 1 Here

To better understand this distinction, consider the production function represented in Figure 1. If, in the short run, the firm must employ fixed capital of $\bar{K}_2$, it can increase output from Q to Q' only by increasing labor from L_0 to L_3, varying the proportions in which the two factors are used. Returns to variable proportions may thus be usefully thought of as describing how output behaves as we move out a *horizontal* through the isoquant map, such as the horizontal $\bar{K}_2AB$. By contrast, consider the input combination (L_1, K_1), producing output level Q at point C. If both capital and labor are doubled, tripled or scaled by any common factor t, the proportions in which they are combined remain unchanged, i.e., capital per worker remains constant at $K_1/L_1 = tK_1/tL_1$ for all $t > 0$, but the production point moves in or out the ray OCD. Thus, we may think of returns to scale as describing how output behaves as we move out through the isoquant map along a *ray* from the origin such as OCD.

In general, returns to scale may be increasing, decreasing, or constant, as output increases more than in proportion, less than in proportion, or exactly in proportion to any change in the scale of input use. In this paper, we shall be concerned exclusively with constant returns to scale. Moreover, we will restrict our attention even further to only those constant returns production functions whose isoquants have the familiar convex-away-from-the-origin shape most commonly encountered in theory and applications. For future reference, we collect up our assumptions and define terms precisely in the following.

DEFINITION 1 *Constant Returns Production*[1]

Let the production function $f(L,K)$ be strictly increasing in L and K, and let its isoquants be strictly convex away from the origin. Then $f(L,K)$ has the property of constant returns to scale (globally) if, for all scalars $t > 0$ and all non-negative input combinations (L,K),

$$f(tL, tK) = tf(L,K).$$

When the production function displays constant returns to scale, doubling both inputs always doubles output; indeed scaling both inputs by any common factor $t > 0$ scales output by exactly that same factor, t. Familiar examples of this sort of production function include the *Cobb-Douglas* form, $Q = AL^aK^{1-a}$, where $A > 0$ and $0 < a < 1$, and the *CES* form, $Q = A(L^r + K^r)^{1/r}$ where $A > 0$ and $0 \neq r < 1$.

Constant returns has significant–and unique–structural implications for the isoquant map. In the next two subsections these are established in a series of propositions, focusing first on the implications of constant returns for the spacing of isoquants looking out a ray from the origin, then looking out a horizontal. In each of these propositions the production function is assumed to satisfy the conditions of Definition 1.

2.1 Looking out a Ray

We begin with a most basic property of the isoquant map under constant returns to scale. We will show that the level of output produced by any combination of inputs will always be proportional to the distance from the origin of the corresponding point in the (L,K)-plane. We recall that a *ray* is any straight line emanating from the origin, and express this property as follows.

[1] A production function having the properties we've assumed is said to be strictly increasing, strictly quasiconcave, and homogeneous of degree one in its arguments. (Note we specifically exclude here linear production functions whose isoquants will be parallel straight lines.) See, for example, Jehle and Reny 2000 for more detail on the mathematical properties of such functions.

Set Figure 2 Here

PROPOSITION 1 *Output is proportional to distance out any ray.*

Proof: In the isoquant map depicted in Figure 2, choose any ray from the origin, OR. Let (L_1, K_1) be the coordinates of the point where OR intersects the isoquant producing one unit of output (the "unit -isoquant"). Let

$$\alpha \equiv \sqrt{L_1^2 + K_1^2} \tag{1}$$

denote the distance along OR from the origin to the unit-isoquant.

Pick any level of output, Q. The point A marks the intersection of the Q-level isoquant and the ray OR. Obviously, Q units of output is Q-times as much as one unit of output. Under constant returns, if L_1 and K_1 together produce one unit then, keeping factor proportions the same, it will take just Q-times as much of each to produce Q units of output. Thus, the coordinates of A must be (QL_1, QK_1), as indicated.

Now compute the distance OA to the Q-unit isoquant along OR, and compare it to the distance to the unit-isoquant along that same ray. According to Pythagorous,

$$\begin{aligned} OA &= \sqrt{(QL_1)^2 + (QK_1)^2} \\ &= \sqrt{(L_1^2 + K_1^2)Q^2} \\ &= \sqrt{(L_1^2 + K_1^2)}Q. \end{aligned}$$

Substituting from (1) into the last line, then solving for Q, we obtain

$$Q = \frac{1}{\alpha} OA.$$

As we sought to show, this says the output produced at any point along a ray from the origin will always be strictly proportional to the distance of that point from the origin. ∎

When output is proportional to distance from the origin, it is easy to see that as long as we remain along the same ray, cutting that distance in half will put you on an isoquant producing only half as much output; doubling or tripling that distance will put you on an isoquant producing twice and three-times as much output, respectively, and so on.

In the next proposition, we consider the distance along a common ray between isoquants giving, successively, equal increments in output.

PROPOSITION 2 *Successive isoquants giving equal increments in output are equally spaced along any ray.*

Proof: Once again refer to Figure 2. Choose any output, Q, and locate its isoquant. Pick an increment in output of any size, $\Delta Q > 0$, and locate the isoquants giving $Q + \Delta Q$ and $Q + 2\Delta Q$ units of output so that the *increment* in output between them is the same, and equal to ΔQ. We need to show that as we look out any ray OR, the distances AB and BC are equal.

We can use the previous result to prove this one. Since output is proportional to distance out the ray OR, we know from Proposition 1 that

$$\begin{aligned} OC &= \alpha(Q + 2\Delta Q) \quad \text{and} \\ OB &= \alpha(Q + \Delta Q) \quad \text{and} \\ OA &= \alpha Q. \end{aligned}$$

Subtracting the second from the first, and the third from the second gives,

$$\begin{aligned} OC - OB &= \alpha(Q + 2\Delta Q - Q - \Delta Q) = \alpha \Delta Q \quad \text{and} \\ OB - OA &= \alpha(Q + \Delta Q - Q) = \alpha \Delta Q, \end{aligned}$$

so $OC - OB = OB - OA$. But $BC = OC - OB$ and $AB = OB - OA$, so $AB = BC$ and our proof is complete. ∎

Propositions 1 and 2 provide useful geometric groundwork for what lies ahead, but our next proposition is the first with direct economic importance. Students will have learned that the (absolute value of) the slope of an isoquant at any point in the (L, K)-plane is called the *marginal rate of technical substitution*, or $MRTS$. The $MRTS$ measures (locally) the rate at which labor can be substituted for capital with no change in the level of output produced, and is therefore important in many decisions the firm must make. In general the $MRTS$ will depend on both L and K separately—if either, or both, are changed we generally expect the $MRTS$ to change as well. Under constant returns, however, the $MRTS$ is *completely independent of scale*, and depends only on factor proportions—whether producing one unit or one million, as long as the firm uses the same amount of capital per worker its possibilities for substituting one for the other remain unchanged.

The implications of this for the isoquant map are sweeping. First recall that only by moving along a ray from the origin in the (L, K)-plane will capital per worker remain constant as scale is varied. If the $MRTS$ depends only on factor proportions, not on scale, then the slope of every isoquant as it crosses a common ray from the origin must always be the same. Though the $MRTS$ will be different along different rays, changing the overall scale of production by moving in or out any given ray will have no effect at all on the $MRTS$. Thus, under constant returns, isoquants must all be *parallel* as we look out any ray. We will now establish this important implication of constant returns production.

PROPOSITION 3 *Isoquants are radially parallel.*

Proof: Consider Figure 3. There we've identified isoquants for arbitrary levels of output Q and Q', and have chosen an arbitrary ray from the origin, OAA'. Ultimately, we want to show that the slope of the tangent to the isoquant at A is equal to the slope of the tangent to the isoquant at A'.

Our approach will be somewhat indirect. To preview, first we construct another ray, OBB', and the chords $A'B'$ and AB. We show that the chord AB is parallel to the chord

$A'B'$, and then we make a limiting argument to complete the proof.

Set Figure 3 Here

To begin, define $\lambda \equiv Q'/Q$, so we may write $Q' \equiv \lambda Q$. Under constant returns, to achieve the λ-fold increase in output between A and A' requires that the amount of labor and capital at A' be exactly λ-times as much as at A. Similarly, the coordinates of B' must be λ-times as great as the coordinates at B. The coordinates of these points are therefore marked accordingly in Figure 3.

It is easy to see that the slope of chord AB is -1 times the ratio AC/CB, or,

$$\text{Slope of the chord } AB = \frac{-(K_A - K_B)}{L_B - L_A}. \tag{2}$$

Similarly, the slope of the chord $A'B'$ is -1 times the ratio $A'C'/C'B'$, or,

$$\begin{aligned}\text{Slope of the chord } A'B' &= \frac{-(\lambda K_A - \lambda K_B)}{\lambda L_B - \lambda L_A} \\ &= \frac{-\lambda(K_A - K_B)}{\lambda(L_B - L_A)} \\ &= \frac{-(K_A - K_B)}{L_B - L_A}. \end{aligned} \tag{3}$$

The right-hand sides of (2) and (3) are the same, so we've shown that the slope of $A'B'$ is equal to the slope of AB.

Now, the slope of $A'B'$ approximates the slope of the tangent at A', and the slope of AB approximates the slope of the tangent at A. To complete our argument, imagine picking the ray OBB' closer and closer to the ray OAA'. The slope of $A'B'$ and AB remain equal to one another as in equations (2) and (3). At the same time, the slope of $A'B'$ becomes a better and better approximation to the slope of the tangent at A', and the slope of AB becomes a better and better approximation to the slope of the tangent at A. In the limit, as the ray OBB' swings toward OAA', the slope of $A'B'$ converges to the slope of the isoquant at A, and the slope of AB converges to the slope of the isoquant at A. But because the chords remain parallel as they approach their respective limits, those limits must be equal, too, so the slope of the tangent at A' must equal the slope of the tangent at A, and our proof is complete. ∎

Notice the high degree of generality in our proof. The two isoquants were picked arbitrarily, and so was the ray OAA'. Therefore, we can be sure that the slopes of *any* two isoquants will be equal along *any* common ray from the origin under constant returns to scale.

2.2 Looking out a Horizontal

In the short run, a firm must operate with fixed amounts of some factor. Just how output behaves as more of the variable factor is combined with that fixed factor has a direct and important impact on the cost of output in the short run.

In the typical textbook example of production in the short run, the total product curve first rises at an increasing rate, then rises at a decreasing rate as output expands—that is, the production function first displays increasing marginal returns and then diminishing marginal returns to the variable factor.

Yet this will not be the behavior of marginal returns when production exhibits constant returns to scale. It is a little-emphasized and perhaps not widely appreciated property of constant returns that, as long as isoquants are strictly convex, both the marginal product of labor and the marginal product of capital are everywhere diminishing—i.e., there is no region in which marginal returns to either factor are either increasing or constant.

For what follows, we recall that the marginal product of a factor (MPL or MPK) is the increment in output produced by a one-unit increase in the amount of that factor used, holding the amount of the other factor constant. We say there are *diminishing marginal returns* to a factor whenever successive one-unit increments in that factor produce smaller and smaller corresponding increments in output. Of course, this is equivalent to saying that to produce equal increments in output requires successively larger and larger increments of the factor. It is this latter interpretation which is most useful in our proof.

PROPOSITION 4 *Marginal returns to both factors (MPL and MPK) are everywhere diminishing.*

Proof: The arguments required to establish these claims with respect to the separate factors labor and capital are completely identical, so we will only give the arguments for labor here, leaving the reader to supply them for the other factor, capital.

First, consider Figure 4. There we've chosen an arbitrary level of output, Q, and an arbitrary increment, ΔQ. Let capital be fixed at $\bar{K}$. To prove this proposition, we must show that under constant returns to scale, the horizontal distance between these isoquants giving equal increments in output becomes larger as we move out that horizontal. In Figure 4, we need to show that $AB < BC$.

Set Figure 4 Here

First construct the ray $OHBDR$, and lines tangent to the isoquants at H and at D. Now look at the triangles $\triangle A'BH$ and $\triangle EBD$. Clearly, $\angle A'BH = \angle EBD$, as these are opposite angles formed by intersecting straight lines. By Proposition 2, $BH = BD$ because these are distances along a common ray between isoquants giving equal increments in output. By Proposition 3, the slope of the isoquant at D must be equal to the slope of the isoquant at H, so side $A'H$ is parallel to side ED. Therefore, as $OHBDR$ cuts those two parallel lines, we must have $\angle BHA' = \angle BDE$.

We've now established that two corresponding angles and the sides including them are equal in $\triangle A'BH$ and $\triangle EBD$. By "angle-side-angle," these two triangles must therefore be *congruent*. From this we conclude that $A'B = BE$, because these are corresponding sides of congruent triangles.

Now notice that with isoquants convex away from the origin, we must also have $AB < A'B$ and $BE < BC$. Putting this all together we get

$$AB < A'B = BE < BC,$$

so $AB < BC$, as we wanted to show. ∎

Notice once again the very general nature of this result: because our choices of output level, increment in output, and level of the fixed factor all were arbitrary, we can be confident that these results apply in every region of the technology.

3 Production and Costs

Properties of the production function have their greatest influence on firm behavior through the impact they have on costs, both short-run and long-run. The very special properties of constant returns production studied in the previous section have stark implications for firm costs.

To begin with the short run, recall that when factor prices are fixed, wherever the production function displays increasing, decreasing, or constant marginal returns to the variable factor, the firm will experience decreasing, constant and increasing short-run marginal costs in the corresponding regions of output. In the typical textbook illustration of this, short-run marginal cost curve first declines at low levels of output then, at higher levels of output, in regions of the technology which begin to display diminishing marginal returns, the marginal cost curve begins to rise.

Though "U-shaped" short-run marginal cost curves may be typical in textbooks, under constant returns to scale production this is definitely *not* how we should expect them to look. Given what we established in Proposition 4, it follows directly that, instead, short-run marginal cost will be *everywhere upward sloping* whenever production displays constant returns to scale. Illustrated in Figure 5, this is important enough to mention in the form of a proposition.

Set Figure 5 Here

PROPOSITION 5 *Short-run marginal cost is everywhere increasing.*

Proof: Suppose capital is fixed in the short run. Then output can increase only if the amount of labor the firm uses is increased. If the (fixed) wage of labor is $w > 0$, then short-run marginal cost at any level of output, defined as the rate of change of short-run total cost at that level of output, will equal the wage cost of the additional labor necessary to produce an incremental unit of output. This allows us to write,

$$\begin{aligned} SMC &= \frac{\Delta STC}{\Delta Q} \\ &= w\Delta L/\Delta Q \\ &= w\frac{1}{\Delta Q/\Delta L} \\ &= w/MPL. \end{aligned}$$

Here, the first line is the definition of SMC, the second line follows from our argument preceding the display, and the third line is a simple re-arrangement of the one preceding it. Note, however, that the denominator in that third line is the rate of change in output as the amount of labor is changed, holding capital constant. That, of course, is just the definition of the marginal product of labor, MPL, so the last line results.

In all of this, we have done nothing more than present the well-known relationship between short-run marginal cost and the marginal product of labor that holds for any production function. But now suppose that production exhibits constant returns to scale. Then by Proposition 4 the marginal product of labor always diminishes as labor, and output, are increased. It is easy to see in the display, above, that SMC must then be always increasing as output rises. ∎

In the long run, no factor is fixed and the profit-maximizing firm chooses amounts of both labor and capital to minimize the cost of producing any level of output. The long-run total cost of output is then simply the cost of the cost-minimizing combination of inputs capable of producing that given level of output.

Set Figure 6 Here

In the common textbook illustration, the solution to the firm's cost-minimization problem is illustrated by the familiar tangency between the relevant isoquant and the lowest *isocost curve* the firm can achieve while still producing the level of output in question, as illustrated in Figure 6[2]. There, all input combinations capable of producing Q units of output lie along the Q-level isoquant. Facing fixed factor prices $w > 0$ and $r > 0$, suppose that all input combinations costing the firm C_0 dollars lie along the isocost curve BA with constant slope $-w/r$; and that all input combinations costing $C_1 > C_0$ dollars lie along the isocost curve $B'A'$, also with constant slope $-w/r$. Then in Figure 6, the input combination (L_0, K_0) both produces output Q and achieves the lowest possible isocost curve, solving the firm's cost-minimization problem for output level Q. The cost of that input combination is therefore the long-run total cost of output Q.

There is a very close relationship, indeed, between scale properties of the technology and the behavior of long-run average cost (LAC), and students find these relationships quite intuitive. Under increasing and decreasing returns, the familiar LAC curve will be upward-sloping and downward-sloping, respectively, while under constant returns to scale the LAC

[2]When the firm faces fixed factor prices $w > 0$ and $r > 0$, the C-dollar isocost curve is the locus of points in the (L, K) plane satisfying $C = wL + rK$. Rearranging, this implies that along that isocost curve, $K = (C/r) - (w/r)L$. When graphed in the (L, K) plane this will be a straight line with slope $-w/r$, vertical intercept C/r and horizontal intercept C/w.

curve will be everywhere horizontal in the output-cost plane. Since our focus in this paper is on constant returns, we will content ourselves with establishing only the relation between constant returns and long run average costs. The cases of increasing and decreasing returns can be established by adapting (though not by simply mimicking) the proof to be given below, and we will leave that as a challenge for the interested reader. For now, we have the following important result.

PROPOSITION 6 *Long-run average cost is constant.*

Set Figure 7 Here

Proof: Suppose the production function depicted in Figure 7 has constant returns to scale. There we have identified the unit-isoquant, giving input combinations capable of producing one unit of output, and we have selected an arbitrary isoquant giving all input combinations capable of producing an output level of Q units.

Suppose that at factor prices $w > 0$ and $r > 0$ the cost of one unit of output is minimized by the input combination (L_1, K_1) at point A. We may then express the long-run total cost (LTC) of one unit of output as

$$LTC(1) = wL_1 + rK_1. \tag{4}$$

To find the input combination that minimizes the cost of Q units of output, we need to find the point where an isocost curve parallel to the one through A is just tangent to the Q-level isoquant. According to Proposition 3, isoquants are radially parallel, so points of equal slope on any two isoquants are always to be found along the same ray from the origin.[3] Hence, the input combination (L_Q, K_Q), along the ray OAB, must minimize the cost of Q.

[3]Hence, under constant returns, the output expansion path is always a ray from the origin.

Now recall that, according to Proposition 1, obtaining a Q-fold increase in output by moving out the same ray requires exactly a Q-fold increase in each factor. Therefore, we must have $L_Q = QL_1$ and $K_Q = QK_1$. After some algebra, and using (4), we obtain:

$$\begin{aligned} LTC(Q) &= wL_Q + rK_Q \\ &= wQL_1 + rQK_1 \\ &= Q(wL_1 + rK_1) \\ &= QLTC(1). \end{aligned} \tag{5}$$

Equation (5) tells us that the long-run total cost of any output level will be proportional to the cost of the very first unit produced: 10 units will cost 10-times as much, 100 units 100-times as much, and so on.

Now long-run average cost is long-run total cost divided by output. Thus, $LAC(Q) = LTC(Q)/Q$ will be the average cost of Q units of output, and $LAC(1) = LTC(1)/1$ will be the average cost of the first unit of output. Dividing both sides of (5) by Q, and substituting from these definitions, we have:

$$LAC(Q) = LAC(1).$$

In other words, the long-run average cost of *any* output level, Q, is always the same, and is equal to the cost of the very first unit produced. ∎

Set Figure 8 Here

To better visualize what we have established in Proposition 6, consider the long-run average cost curve in Figure 8. In the preceding proof, we picked an arbitrary level of output, Q, and then showed that $LAC(Q) = LAC(1)$, so that the long-run average cost curve must be horizontal, as depicted in Figure 8.

In that same figure, notice that the horizontal curve bears the label LAC, for long-run average cost, *and* the label LMC for long-run marginal cost. You will recall that, according to the well-known relationship between averages and marginals, when the average is constant, the marginal is also constant—and must be equal to the average.[4] It therefore follows immediately from Proposition 6 that under constant returns to scale, long-run marginal cost, LMC, is constant, too, and is equal to long-run average cost. To help remember the important connection between LAC and LMC, we shall conclude by recording, without further proof, this important corollary to the previous proposition.

PROPOSITION 7 *Long-run marginal cost is constant and equal to long-run average cost.*

4 Conclusion

In this short paper we have used simple geometry to establish several crucial properties of production and cost under constant returns to scale, both in the short run and in the long run. All of these results are important, and all bear careful study because the very special properties of constant returns production are central to so many important results in theory and the applied fields.

[4] "Marginals" measure change in their associated "total." If the average is rising, the margin, or that being added to the total, must be above the average, pulling the average up; if the average is falling, the margin must be below the average, pulling the average down. If the average is unchanging, then the margin must be neither above nor below the average–it must equal the average.

Mathematically, let $T(x)$ be any total measure (e.g. total cost, total product, total profit). Then let $M(x)$ be the associated marginal measure and $A(x)$ the associated average measure. By definition, $M(x) \equiv T'(x)$ and $A(x) \equiv T(x)/x$, so we can write $T(x) \equiv xA(x)$. Differentiating both sides of this identity with respect to x (remember to use the chain rule on the rhs), substituting from the definition of $M(x)$, and rearranging we can express the slope of the average curve at any point, $x > 0$, as follows:

$$A'(x) = \frac{M(x) - A(x)}{x}.$$

Note that for any $x > 0$, $A'(x) > 0$ ($A'(x) < 0$) if and only if $M(x) > A(x)$ ($M(x) < A(x)$). Similarly, $A'(x) = 0$ (i.e., the average curve is flat) if and only if $M(x) = A(x)$ (i.e., the marginal and average curves coincide.)

Yet even the list of topics we've examined here is far from exhaustive—there is more one can discover about constant returns production. Soper 1967, for example, using little more geometry than we've deployed here, presents a cogent and very accessible geometric proof of Euler's Theorem, so central to the famous "product exhaustion theorem" of competitive economics.

Finally, it should be noted that many of the results presented here in the context of constant returns production have direct analogies in the theory of consumer demand under homogeneous (or, indeed, homothetic) utility. By simply reinterpreting, and occasionally extending, the principles established here, the reader should be able to explore, alone, those closely related neighborhoods of economic theory.

References

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Figures

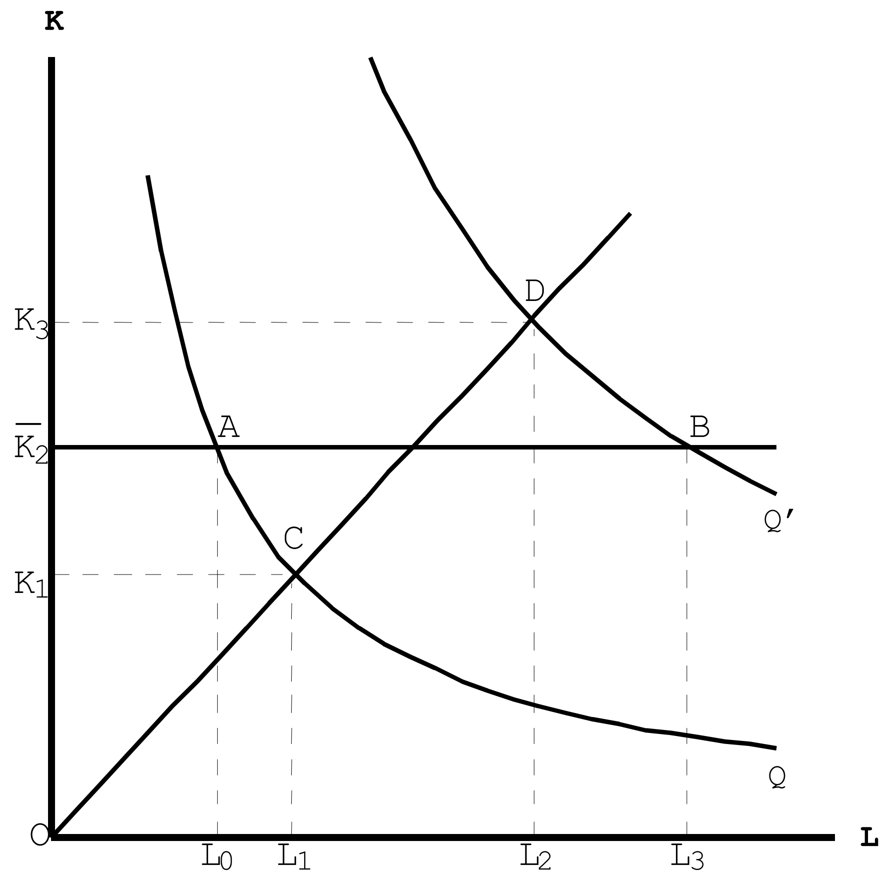

Figure 1

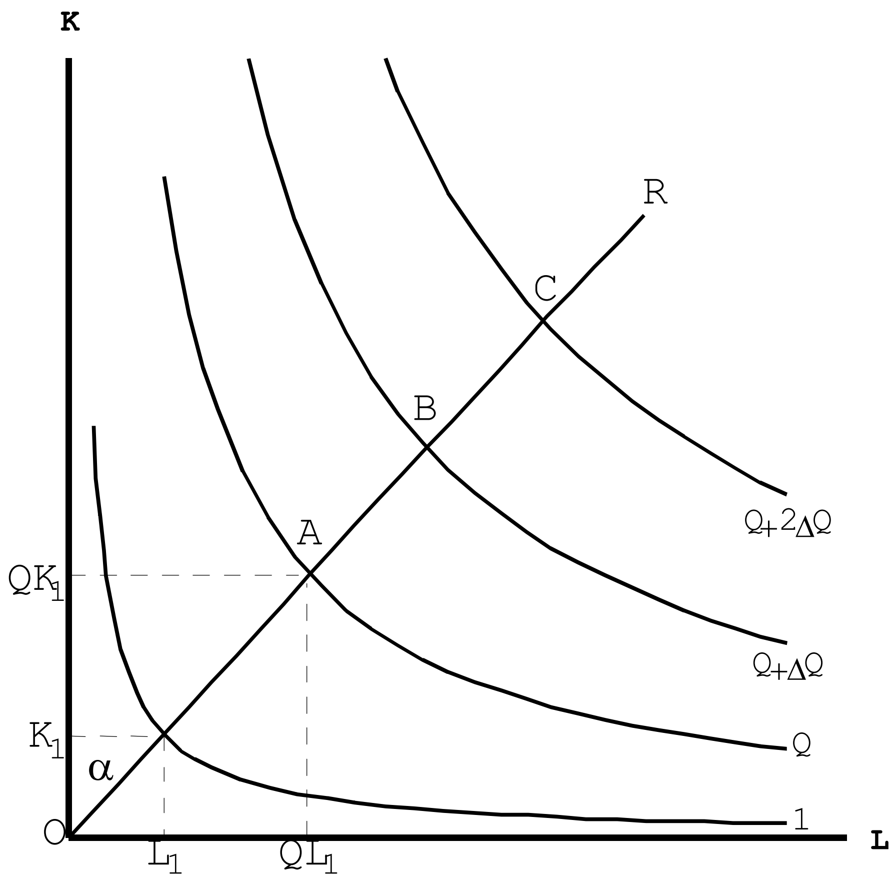

Figure 2

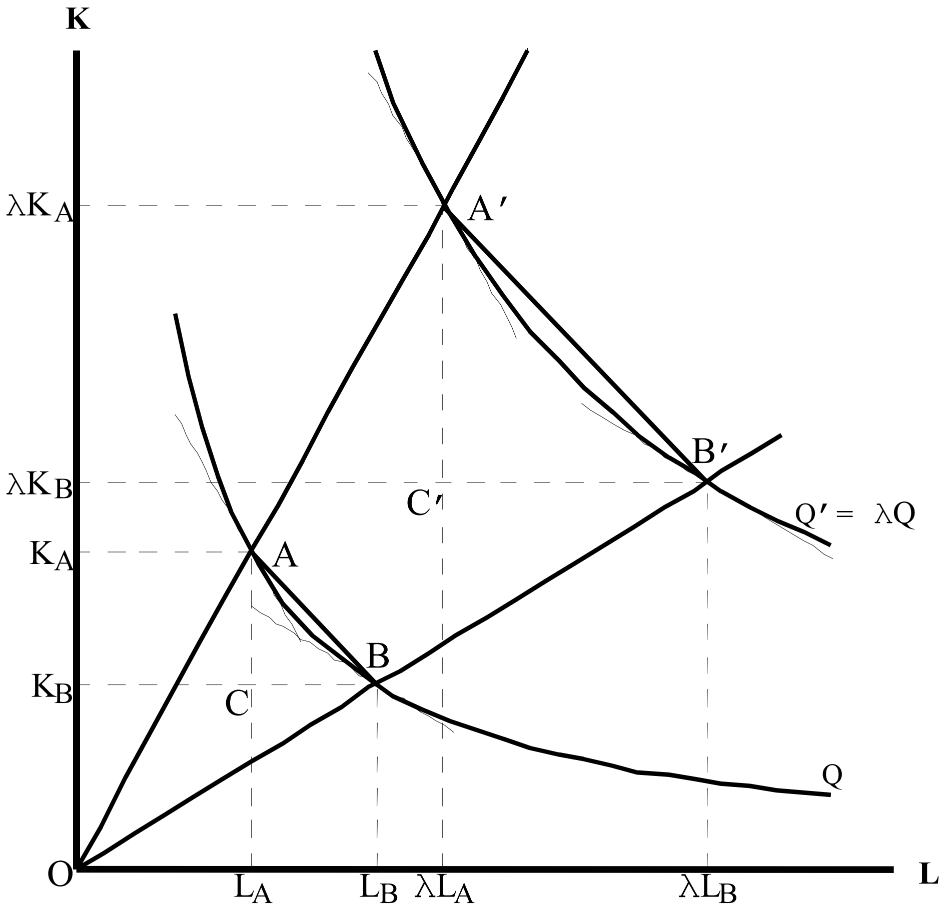

Figure 3

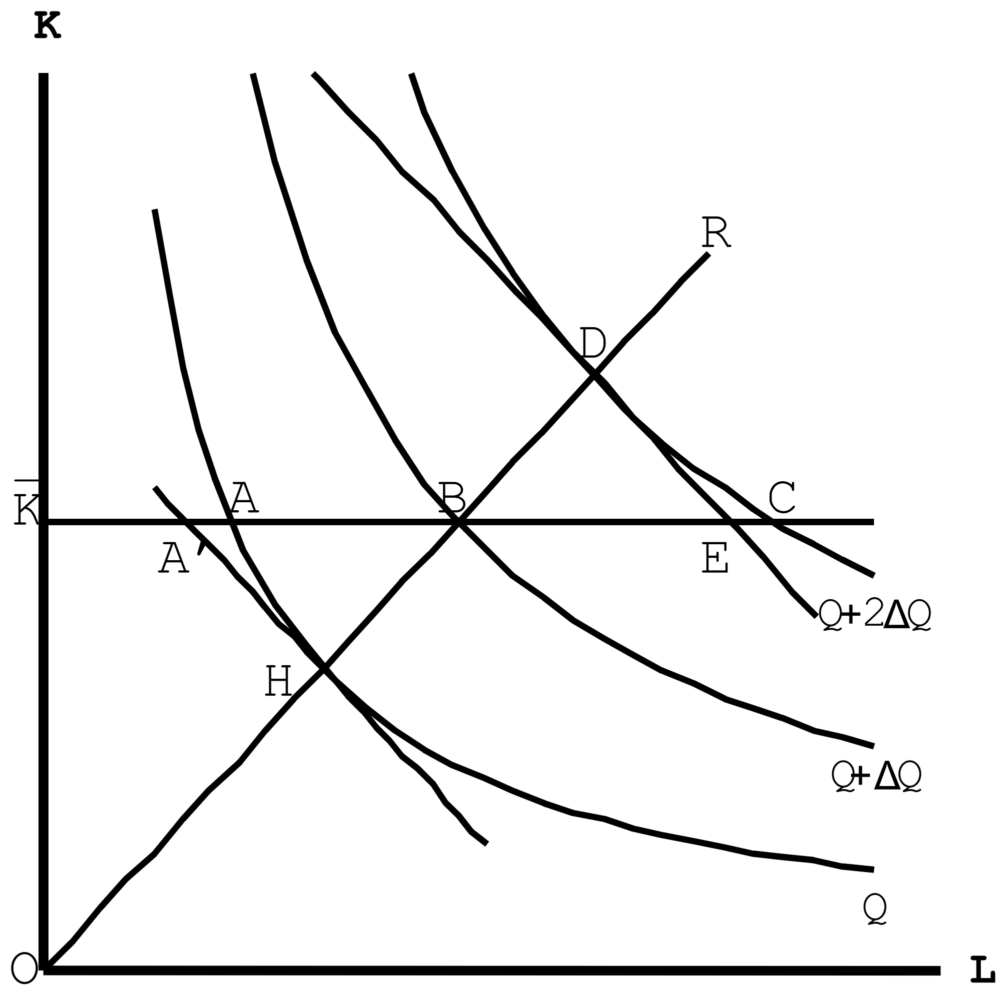

Figure 4

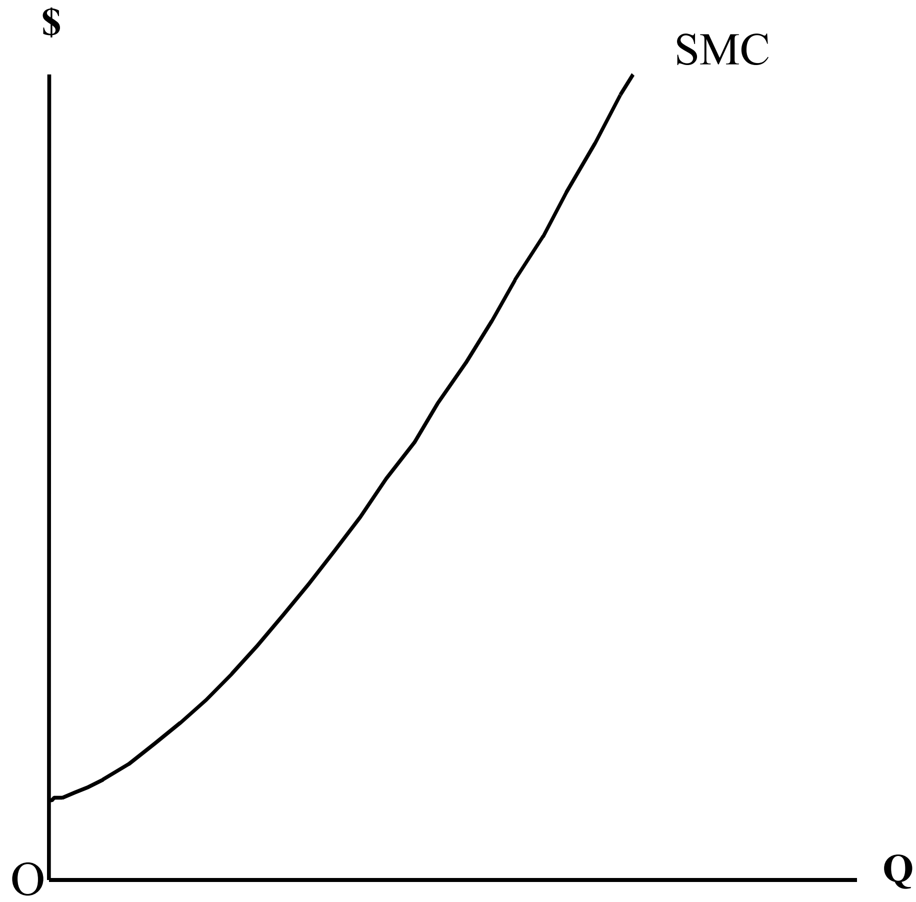

Figure 5

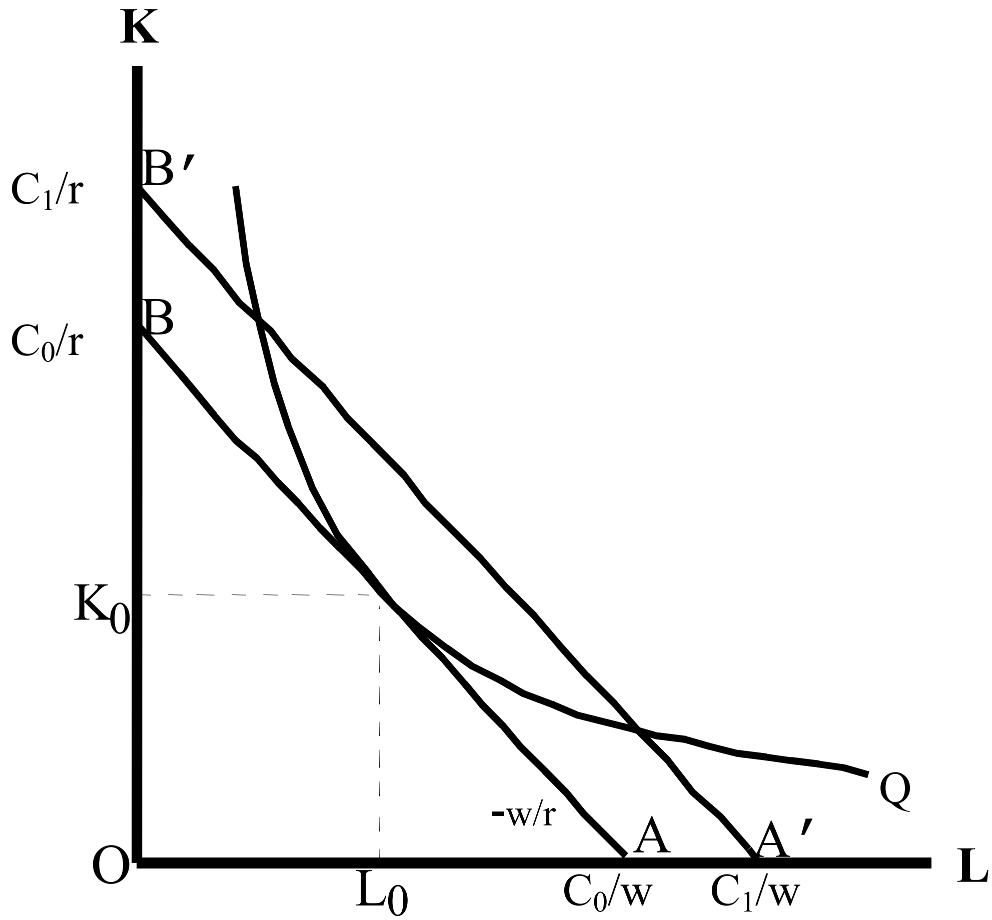

Figure 6

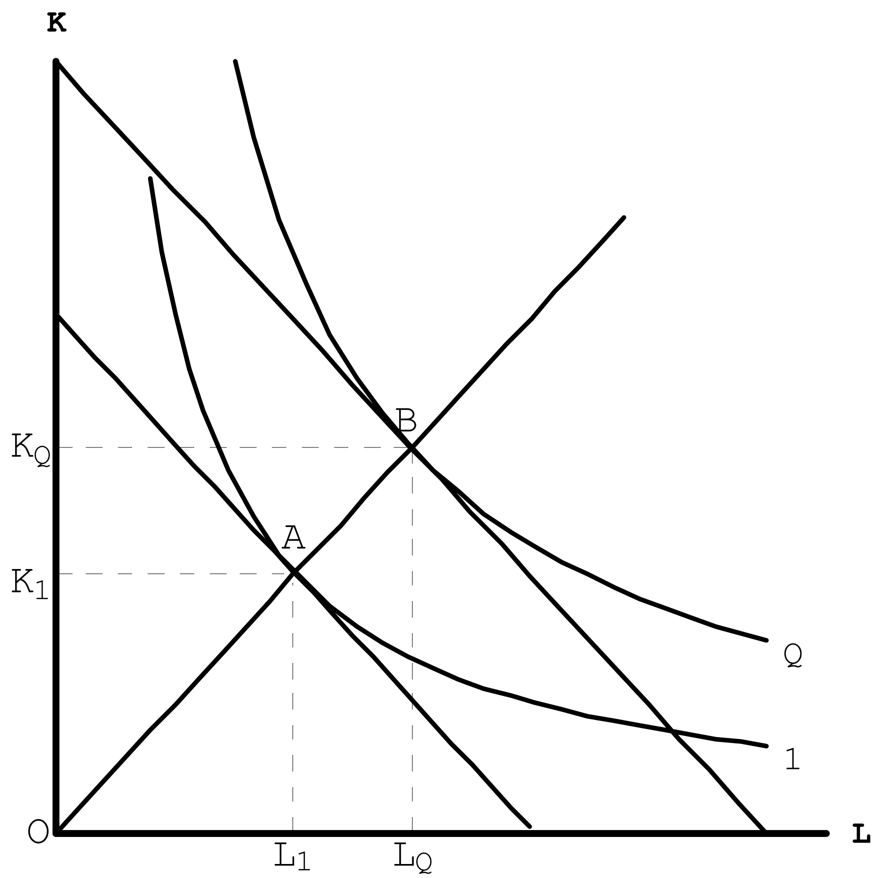

Figure 7

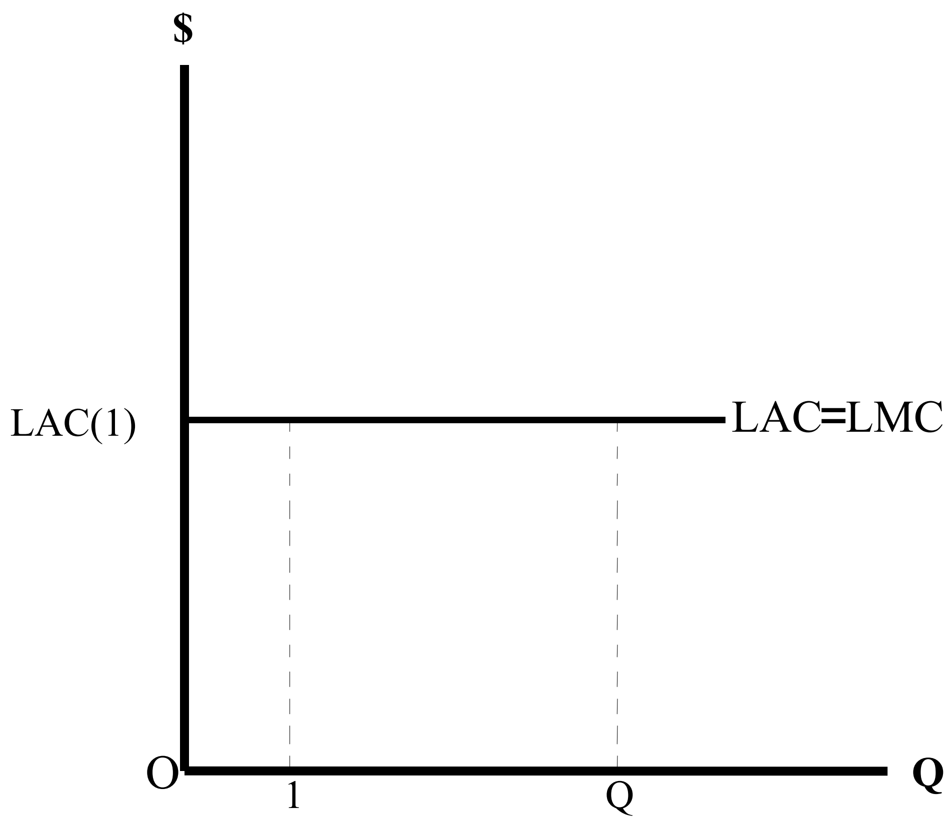

Figure 8