NEAR EXTREMAL KHOVANOV HOMOLOGY AND 4-GENUS OF TURAEV GENUS ONE LINKS

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BACKGROUND

INTRODUCTION TO KNOT THEORY

A **knot** is technically defined as an equivalence class of closed curves in 3-space but think of it as a string tied with the ends glued together.

A **knot diagram** is the projection of a knot into the plane.

Two diagrams represent the same knot if one diagram can be deformed (via isotopy) into the other without cutting any of the strands.

Knot invariants are properties that remain the same between different diagrams of the same knot and can be used to differentiate knots and determine properties of knots.

SOME TYPES OF KNOTS & CROSSINGS

A knot is **alternating** (Figure 1a) if it has an alternating diagram, i.e. if you follow along a strand it alternates between going over and under when it crosses another strand.

A knot is **almost-alternating** (Figure 1b) if it has no alternating diagram, and it has a diagram where if one crossing was changed it would be an alternating diagram. That crossing is called the dealternator and is circled in red in the figure below.

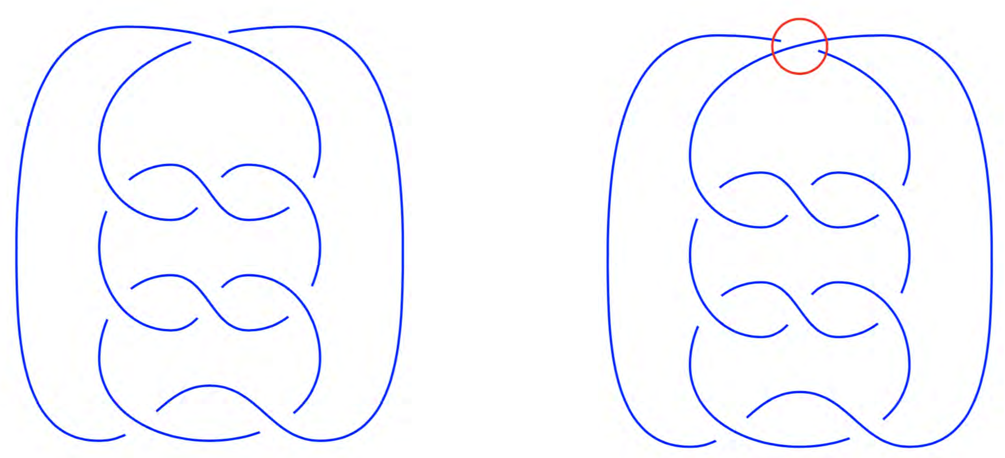

Figure 1: An alternating (a) and an almost alternating (b) knot

Giving a knot **orientation** allows us to see whether its crossings are **positive** (Figure 2a) or **negative** (Figure 2b). The decision that the crossing on the left is positive is convention—all theorems regarding positive crossings have corresponding "mirror" theorems for negative crossings.

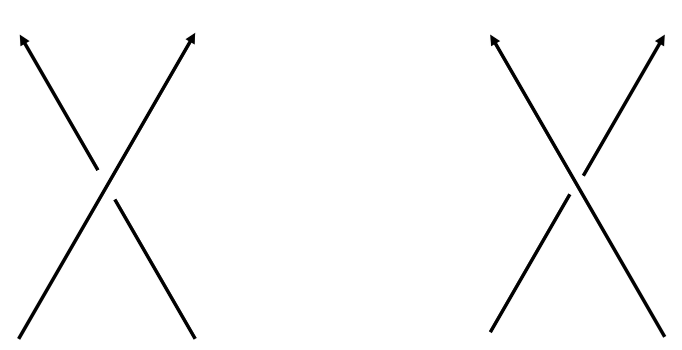

Figure 2: A positive (a) and a negative (b) crossing

TURAEV GENUS ONE/ALMOST ALTERNATING TEST

A knot is **Turaev genus one** if it has the form of the knot in Figure 3 below, where the interiors of the blue circles vary, and the number of circles varies by a factor of 2. Within the blue circles the knot is alternating, and between the circles it is not. Every Turaev genus one knot is either almost-alternating or is a type of knot with similar properties called semi-adequate.

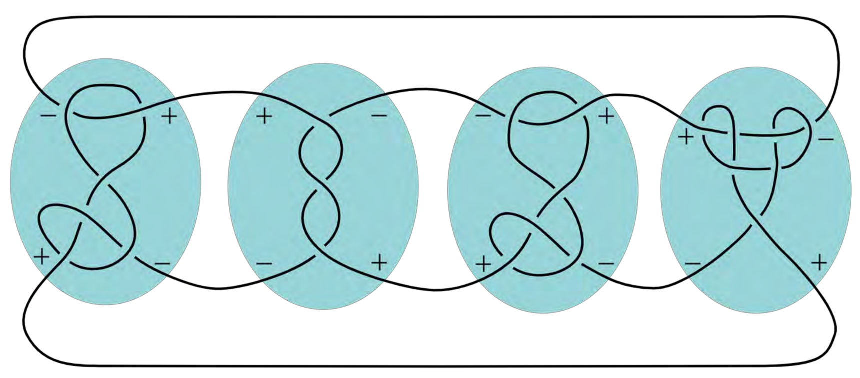

Figure 3: A Turaev genus one knot

Khovanov homology is an algebraic knot invariant, represented by a grid of numbers where the dimension of the homology group Kh(i,j) is the value of the box in the ith column and jth row.

Our main theorem is the following:

THEOREM 1

If a knot K is Turaev genus one or almost alternating, then either

$Kh(i_{min}, j_{min})(K) = 1$ and $Kh(i_{min} + 2, j_{min} + 2)(K) = 0$, or

$Kh(i_{max}, j_{max})(K) = 1$ and $Kh(i_{max} - 2, j_{max} - 2)(K) = 0$.

The corresponding cells are highlighted in orange in Figure 6.

RESULTS OF THEOREM 1

All Turaev genus one/almost alternating diagrams have a Khovanov homology with this property.

Test to find Turaev genus one/almost alternating knots: If the Khovanov homology of a knot doesn't have the property the theorem predicts, then it is not Turaev genus one.

Ex. Our theorem allows us to determine that the knot pictured in Figure 4 is not Turaev genus one, which was previously unknown.

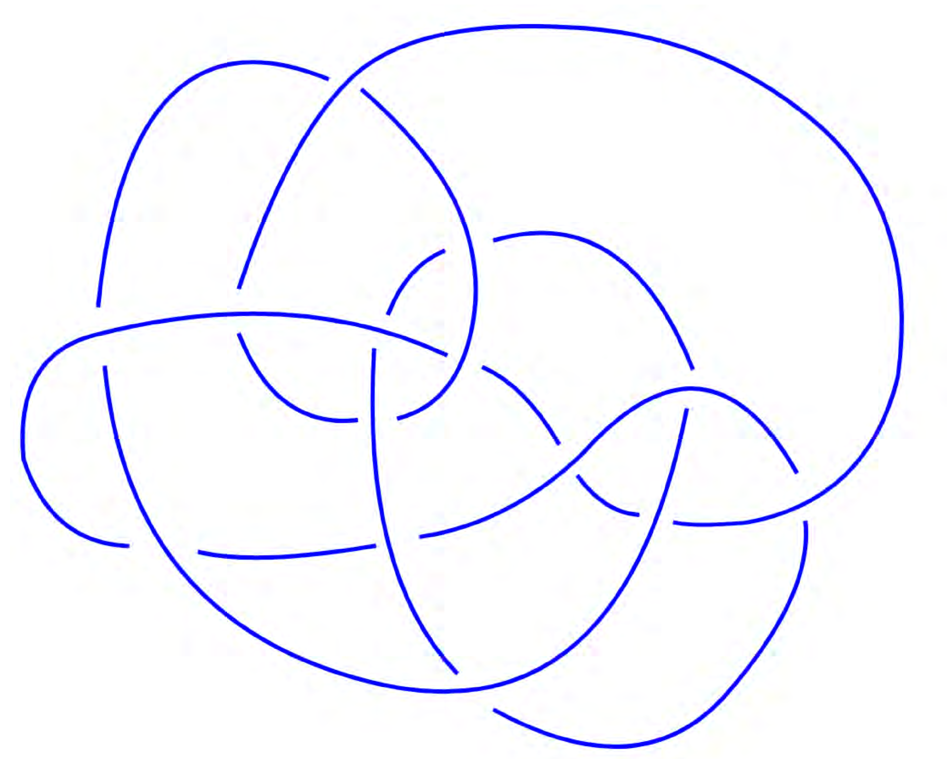

Figure 4: A non-almost alternating knot

j\i	-2	-1	0	1	2	3	4	5	6	7	8
19											2
17										1	
15									2	2	
13							1	3	1		
11							1	2			
9					1	4	3				
7				1	1	1					
5				1	3						
3		1	2	1							
1			1								
-1	1										

Figure 5: Khovanov homology for the knot in Figure 4. Note that it does not satisfy the conditions in Theorem 1.

4-GENUS OF TURAEV GENUS ONE KNOTS

The 4-genus of a knot is the fewest number of holes in any surface in 4- space that is bounded by the knot.

THEOREM 2

If K is Turaev genus one and satisfies an additional condition on the number of negative crossings, then there exists a quantity s(K) such that ½ |s(K)| ≤ g_4 ≤ ½ |s(K)| +1.

Using this theorem, we can calculate the possible values of the 4-genus for many Turaev genus one knots.

For example, for the Turaev genus one knot 9_20 (pictured below), ½|s(K)| = ½|c - s_A(K) - 2| = ½|12 - 6 - 2| = ½|4| = 2 and we know g_4(K)=2. Thus our theorem correctly predicts 2 ≤ g_4(K)≤ 3.

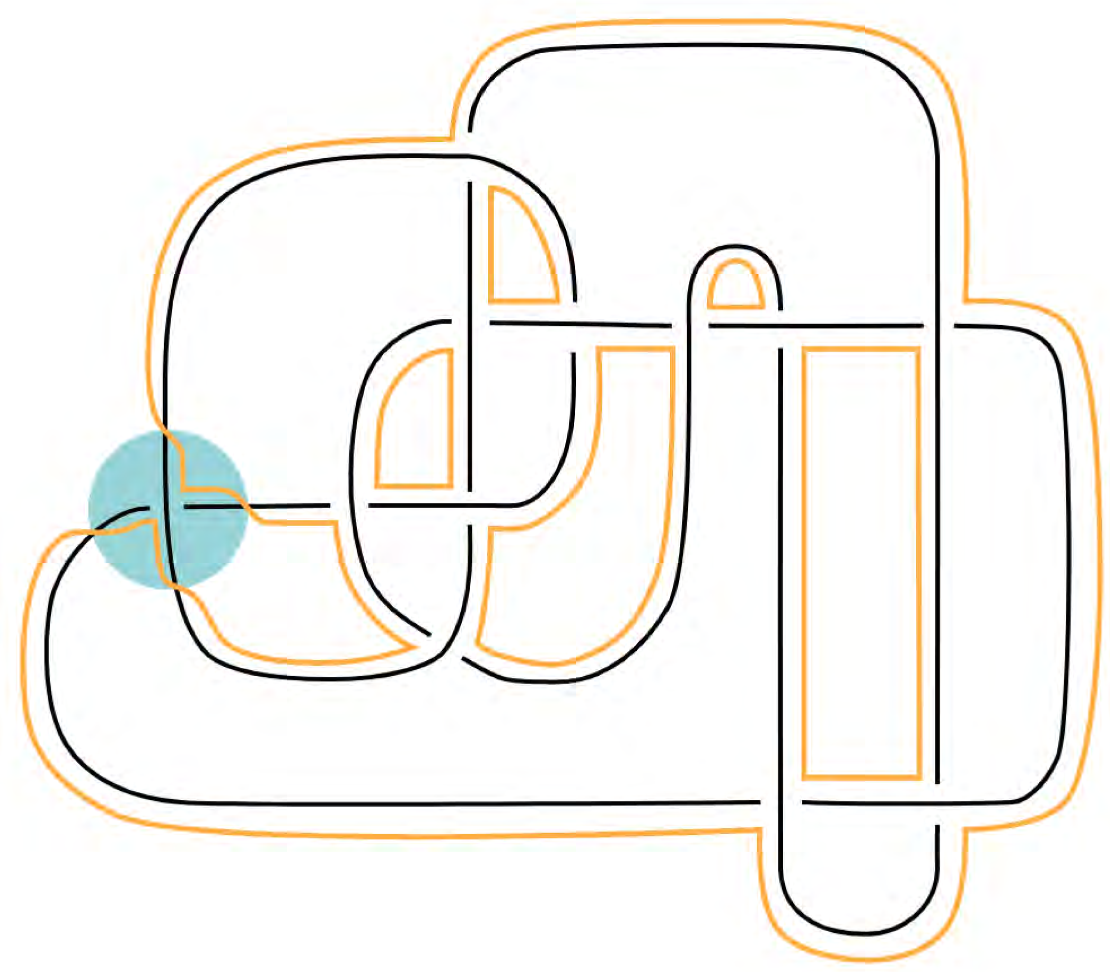

Figure 6: A Turaev genus one knot. The number of orange shapes is s_A(K) in the computation above, c represents the number of crossings in the diagram, and the blue circle highlights the dealternator.

ACKNOWLEDGEMENTS + REFERENCES

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